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Asymptotic structure of perturbative series for τ lepton decay observables: m_s^2 corrections

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Abstract

In a previous paper [1] we performed an analysis of asymptotic structure of perturbation theory series for semileptonic τ-lepton decays in massless limit. We extend our analysis to the Cabibbo suppressed $\Delta S = 1$ decay modes of the τ lepton. In particular we address the problem of m_s^2 corrections to theoretical formulas. The properties of the asymptotic behavior of the finite order perturbation theory series for the coefficient functions of the m_s^2 corrections are studied.

1 Introduction

The accuracy of experimental data for τ lepton decays makes it feasible now to extract the spectral density of the Cabibbo suppressed $\Delta S = 1$ decay modes through detecting strange hadrons and even to pin down the tiny difference with the Cabibbo favored $\Delta S = 0$ case due to the non-vanishing strange quark mass [2, 3]. One of the main problems in obtaining precise theoretical formulas is the strict control over the convergence of perturbation theory (PT) series and the error due to its truncation [1]. For the nonstrange decay channels (Cabibbo favored) this problem is now an actual problem – the theoretical uncertainty has already reached a limiting value existing due to asymptotic nature of the PT series. This value is comparable in magnitude with the experimental error. For the Cabibbo suppressed modes the experimental errors are still larger than theoretical uncertainties. However, with the accuracy of experimental data permanently improving the limiting theoretical precision within FOPT is becoming a major problem of theoretical analysis in general and of the extraction of the strange quark mass m_s from m_s^2 corrections in particular.

From the theoretical point of view one of the central quantities of interest for the Cabibbo suppressed modes from theoretical point of view is the correction to hadronic spectral density arising from the nonvanishing s-quark mass. This makes the description different compared to the massless (Cabibbo favored or ud) case. The m_s^2 corrections to the spectral densities have been calculated with a high degree of accuracy within perturbation theory in the strong coupling constant (e.g. [4]).

In the present note we determine the ultimate theoretical precision reachable for m_s^2 corrections within a finite order perturbation theory analysis. We follow closely the lines of ref. [1] and reach our conclusions in a renormalization scheme invariant way.

The basic observable is the normalized τ lepton decay rate into hadrons written in the standard form

$$R_\tau = \frac{\Gamma(\tau \to h\nu)}{\Gamma(\tau \to l\nu\bar{\nu})} = N_c S_{EW}((|V_{ud}|^2(1+\delta_{ud}) + |V_{us}|^2(1+\delta_{us}))\,. \tag{1}$$

The leading terms in eq. (1) are the parton model results while the terms δ_{ud} and δ_{us} represent the effects of QCD interaction and (in case of nonvanishing quark masses) mass effects [5, 6, 7, 8, 9]. V_{ud} and V_{us} are matrix elements of the weak mixing matrix and S_{EW} describes the electroweak radiative corrections to the τ-decay rate.

In general, hadronic observables in the τ system are related to the two-point correlator of hadronic currents with well established and simple analytic properties – this makes the comparison of experimental data with theoretical calculations very clean. This feature makes τ lepton physics an important area of particle phenomenology where theory (QCD) can be confronted with experiment to a level of very high precision.

The correlator (here we concentrate only on strange hadronic current, i.e. the term proportional to V_{us}) has the form

$$\Pi_{\mu\nu}(q) = 12\pi^2 i \int dx e^{iqx} \langle T j_\mu(x) j_\nu^\dagger(0) \rangle = q_\mu q_\nu \Pi_q(q^2) + g_{\mu\nu}\Pi_g(q^2) \tag{2}$$

with $j_\mu(x) = \bar{u}\gamma_\mu(1-\gamma_5)s$. $\Pi_q(q^2)$ and $\Pi_g(q^2)$ are invariant scalar functions. We work within QCD with three light quarks and do not consider corrections due to heavy quarks (c-quark) that would enter in higher orders of PT through loop effects [10]. The correlator is normalized to unity in the leading parton model approximation with massless quarks.

The theoretical expression for the QCD part of the decay rate into strange hadrons is given by (with $N_c|V_{us}|^2 S_{EW}$ factored out)

$$R_\tau = \int_0^{M_\tau^2} 2\left(1 - \frac{s}{M_\tau^2}\right)^2 \left(R_q(s) - \frac{2}{M_\tau^2} R_g(s)\right) \frac{ds}{M_\tau^2} \tag{3}$$

with $R_q(s)$ and $R_g(s)$ being the absorptive parts of the structure functions $\Pi_q(q^2)$ and $\Pi_g(q^2)$. The masses of the light quarks (u, d) can be neglected. We study m_s^2 corrections for the Cabibbo suppressed decay modes. The representation of the total decay rate in terms of the absorptive parts of the structure functions $\Pi_q(q^2)$ and $\Pi_g(q^2)$ is convenient from the point of view of their analytic properties in the complex q^2-plane. The physical decomposition of

the correlator (2) reads

$$\Pi_{\mu\nu}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi_T(q^2) + q_\mu q_\nu \Pi_L(q^2) \tag{4}$$

where the $\Pi_T(q^2)$ part contains only spin 1 contributions and $\Pi_L(q^2)$ contains only spin 0 contributions. The relation between the two sets of invariant functions $\Pi_{T,L}(q^2)$ and $\Pi_{q,g}(q^2)$ describing the correlator eq. (2) reads

$$\Pi_T(q^2) = \frac{\Pi_g(q^2)}{-q^2}, \quad \Pi_L(q^2) = \Pi_q(q^2) + \frac{\Pi_g(q^2)}{q^2} \ . \tag{5}$$

In terms of the physical (definite spin) invariant functions eq. (3) reads

$$R_\tau = \int_0^{M_\tau^2} 2\left(1 - \frac{s}{M_\tau^2}\right)^2 \left(\left(1 + 2\frac{s}{M_\tau^2}\right) R_T(s) + R_L(s)\right) \frac{ds}{M_\tau^2} \ . \tag{6}$$

The longitudinal part of the spectral density $R_L(s)$ vanishes if all quarks are assumed to be massless. On expanding $\Pi_q(q^2)$ and $\Pi_g(q^2)$ in m_s^2/q^2 and keeping only the leading term in this expansion one has

$$\Pi_q(q^2) = \Pi(q^2) + 3\frac{m_s^2}{q^2}\Pi_{mq}(q^2) \tag{7}$$

$$\Pi_g(q^2) = -q^2\Pi(q^2) + \frac{3}{2}m_s^2\Pi_{mg}(q^2) \tag{8}$$

where $\Pi(q^2)$ is the invariant function already known from the mass zero case. The functions $\Pi_\#(Q^2)$ with $Q^2 = -q^2$ are computable in perturbation theory in the deep Euclidean region $Q^2 \to \infty$. The results of the PT calculation read

$$\begin{aligned}
-Q^2\frac{d}{dQ^2}\Pi(Q^2)\Big|_{Q^2=\mu^2} &= 1 + \frac{\alpha_s}{\pi} + k_1\left(\frac{\alpha_s}{\pi}\right)^2 + k_2\left(\frac{\alpha_s}{\pi}\right)^3 + k_3\left(\frac{\alpha_s}{\pi}\right)^4 + O(\alpha_s^5) \\
-Q^2\frac{d}{dQ^2}\Pi_{mg}(Q^2)\Big|_{Q^2=\mu^2} &= 1 + \frac{5}{3}\frac{\alpha_s}{\pi} + k_{g1}\left(\frac{\alpha_s}{\pi}\right)^2 + k_{g2}\left(\frac{\alpha_s}{\pi}\right)^3 + k_{g3}\left(\frac{\alpha_s}{\pi}\right)^4 + O(\alpha_s^5) \\
\Pi_{mq}(Q^2)\Big|_{Q^2=\mu^2} &= 1 + \frac{7}{3}\frac{\alpha_s}{\pi} + k_{q1}\left(\frac{\alpha_s}{\pi}\right)^2 + k_{q2}\left(\frac{\alpha_s}{\pi}\right)^3 + O(\alpha_s^4)
\end{aligned} \tag{9}$$

Even though the fourth order $\overline{\mathrm{MS}}$ scheme coefficient k_3 and the coefficients k_{q2}, k_{g3} are not known at present we retain their contributions since we want to dispose on them as free parameters for later considerations. The numerical values of the other coefficients in the $\overline{\mathrm{MS}}$

scheme are given in the Appendix: these are needed as reference numbers for our transformations to other more appropriate schemes. The running coupling and mass are renormalized at the scale μ. The light quarks u, d are taken to be massless. Eq. (9) constitutes the complete theoretical information necessary for our fixed order perturbation theory analysis of m_s^2 corrections. The corresponding expressions with an explicit Q^2 dependence can be found by inserting the expansion of the running coupling constant

$$\begin{aligned}\frac{\alpha_s(Q^2)}{\pi} &= \frac{\alpha_s}{\pi} + \beta_0 L(\frac{\alpha_s}{\pi})^2 + (\beta_1 L + \beta_0^2 L^2)(\frac{\alpha_s}{\pi})^3 + (\beta_2 L + \frac{5}{2}\beta_1\beta_0 L^2 + \beta_0^3 L^3)(\frac{\alpha_s}{\pi})^4 \\ &+(\beta_3 L + 3\beta_0\beta_2 L^2 + \frac{3}{2}\beta_1^2 L^2 + \frac{13}{3}\beta_0^2\beta_1 L^3 + \beta_0^4 L^4)(\frac{\alpha_s}{\pi})^5 + \ldots \qquad (10)\end{aligned}$$

and that of the running mass

$$\frac{m_s(Q^2)}{m_s(\mu^2)} = 1 + L\gamma_0(\frac{\alpha_s}{\pi}) + (\frac{1}{2}L^2\beta_0\gamma_0 + \frac{1}{2}L^2\gamma_0^2 + L\gamma_1)(\frac{\alpha_s}{\pi})^2 + ... \qquad (11)$$

Here β_i and γ_i are the appropriate coefficients of β- and γ-functions describing the evolution (running) of the coupling and mass and

$$L = \ln(\frac{\mu^2}{Q^2}). \qquad (12)$$

The coupling constant α_s in eq. (10, 11) in taken at a genuine normalization point μ. In the present note we do not systematically discuss non-perturbative effects stemming from standard power corrections [11]. The standard power corrections arise from nonvanishing vacuum expectation values of local operators within the operator product expansion and are relatively small. They can be simply accounted for if necessary. They do not mix with the m_s^2 corrections. The coefficient functions of the local operators are known in low orders of the perturbative expansion and there is no necessity for a thorough analysis of their convergence properties at present.

2 Natural strange quark masses for internal perturbation theory description of m_s^2 corrections

We restrict our attention only to the new features that appear due to the mass corrections when compared to our previous analysis [1]. The appropriate quantities to consider are moments of a spectral density

$$M_n = (n+1)\int_0^1 \rho(x)x^n dx \ . \tag{13}$$

We often use M_τ as a unit of mass which leads to dimensionless variable $x = s/M_\tau^2$. Note that within finite order perturbation theory the moments eq. (13) coincide with the results of a contour integration [12, 13, 14, 15] because of analytic properties of the functions $\ln^p z$. The moments of the hadronic spectral densities are internal characteristics of the hadronic decays of the τ system and it is instructive to describe these moments in internal variables. In the massless limit within FOPT there is only one independent internal variable – the effective coupling $a(s)$ which is defined directly on the physical cut through the relation

$$\rho(s) = 1 + a(s) \tag{14}$$

and studied in ref. [1]. All the constants that may appear due to a particular choice of the renormalization scheme are absorbed into the definition of the effective charge e.g. [16, 17, 18, 19]. In our present analysis the effective charge is determined by the massless piece $\Pi(q^2)$ of the correlator (2) (see, eq. (7,8)). Eq. (14) fixes the definition of the effective charge which is later used as an expansion parameter for the mass corrections. With such a definition one retains consistency in the description of the massless approximation for strange and non-strange modes.

The running of the coupling $a(s)$ defined in eq. (14) contains logarithms of s with coefficients given by an effective β-function

$$\bar\beta(a) = s\frac{da(s)}{ds} = -\beta_0 a^2 - \beta_1 a^3 - \bar\beta_2 a^4 - \bar\beta_3 a^5 + O(a^6) \tag{15}$$

and reads

$$a(s) = a + \beta_0 l a^2 + (\beta_1 l + \beta_0^2 l^2) a^3 + (\bar\beta_2 l + \frac{5}{2}\beta_1\beta_0 l^2 + \beta_0^3 l^3) a^4$$
$$+(\bar\beta_3 l + 3\beta_0\bar\beta_2 l^2 + \frac{3}{2}\beta_1^2 l^2 + \frac{13}{3}\beta_0^2\beta_1 l^3 + \beta_0^4 l^4) a^5 + \ldots \quad (16)$$

where $a = a(M_\tau^2)$, $l = \ln(M_\tau^2/s)$. Note that the expansion of $a(s)$ in eq. (16) has the same form as the one for $\alpha_s(s)/\pi$ but now $\bar\beta_2$ and $\bar\beta_3$ are coefficients of the effective β-function [1] while β_0 and β_1 are renormalization group invariants. The effective coupling a can be expressed through the $\overline{\rm MS}$ coupling constant

$$a = \frac{\alpha_s}{\pi} + 1.64\left(\frac{\alpha_s}{\pi}\right)^2 - 10.28\left(\frac{\alpha_s}{\pi}\right)^3 + (-155.0 + k_3)\left(\frac{\alpha_s}{\pi}\right)^4 + ... \quad (17)$$

with $\alpha_s \equiv \alpha_s(M_\tau^2)$. The massless spectral density reads [1]

$$\rho(s) = 1 + a + 2.25a^2 l + a^3(4l + 5.063l^2) + a^4(-25.7l + 22.5l^2 + 11.4l^3)$$
$$+a^5((-409.5 + 4.5k_3)l - 149.4l^2 + 87.75l^3 + 25.63l^4) + O(a^6). \quad (18)$$

At any fixed order of perturbation theory the effects of running die out for the high order moments (large n in eq. (13)) improving the convergence of the perturbation theory series. With the definition of the charge according to eq. (14) all high order corrections vanish as $n \to \infty$ for any fixed order of perturbation theory. However, for the mass corrections this is not true anymore since the strange quark mass introduces a new parameter. In order to obtain only logarithms of energy in the spectral density one can redefine the $\overline{\rm MS}$-scheme quark mass appropriately and absorb all remaining constants into the internal mass parameter. Because there are two invariant functions that characterize the correlator (2) the definitions of the mass parameters may be different for them. This redefinition is nothing but the change of subtraction scheme for the given spectral density. Note that the introduction of a natural internal coupling parameter such as the effective charge $a(s)$ allows one to extend the perturbation theory series needed for the description of relations between observables by one more term as compared to the analysis in e.g. the $\overline{\rm MS}$-scheme (e.g. [20, 21, 22, 23]). The

same reason of obtaining one more term in the perturbation series is behind the definition of internal parameters for the quark mass.

2.1 The g-part of the correlator: function $\Pi_{mg}(q^2)$

First we consider the g-part of the correlator which contains spin 1 contributions only. We use the massless case [1] as a base for our analysis here. We define a new mass parameter m_g for the strange quark to absorb all constants in the mass correction of the spectral density into the new mass definition. By definition

$$m_s^2(s)\rho_g^{\overline{\rm MS}}(s;\alpha_s) \equiv m_g^2(s;a) = m_g^2(M_\tau^2)\rho_g(s). \tag{19}$$

The new mass parameter $m_g = m_g(M_\tau^2)$ is related to the $\overline{\rm MS}$-scheme mass parameter $m_s = m_s(M_\tau^2)$ through

$$\begin{aligned} m_g^2 &= m_s^2\left(1+1.67a-5.87a^2-51.0a^3+(-1342.5-1.67k_3+k_{g3})a^4+O(a^5)\right) \qquad (20)\\ &= m_s^2\left(1+1.67\left(\frac{\alpha_s}{\pi}\right)-3.14\left(\frac{\alpha_s}{\pi}\right)^2-87.4\left(\frac{\alpha_s}{\pi}\right)^3+\right.\\ &\quad \left.(-1750+k_{g3})\left(\frac{\alpha_s}{\pi}\right)^4+O\left(\left(\frac{\alpha_s}{\pi}\right)^5\right)\right) \end{aligned}$$

The spectral density $\rho_g(s)$ for the g-part of the mass correction contains only logarithms in fixed order PT expansion and reads

$$\begin{aligned} \rho_g(s) &= 1+2al+a^2(8.05l+4.25l^2)+a^3(5.3l+38.23l^2+9.21l^3)\\ &\quad +a^4((-45.3-2.0k_3)l+67.4l^2+134.2l^3+20.14l^4)+O(a^5). \end{aligned} \tag{21}$$

With the help of eq. (20) the numerical value of m_g^2 can be determined in terms of the $\overline{\rm MS}$-scheme mass squared m_s^2 with an accuracy of about 7% if eq. (20) is evaluated up to third order. If the series is (perturbatively) inverted, the $\overline{\rm MS}$-scheme strange quark mass can be extracted with an accuracy of about 4% from the relation

$$m_s^2 = m_g^2(1-0.185+0.107+0.037+(0.19+0.00025k_3-0.00015k_{g3})). \tag{22}$$

This accuracy is sufficient at present for phenomenological applications.

Given the expression for the spectral density eq. (21) the whole analysis of ref. [1] applies. The moments of the spectral density $\rho_g(s)$ in general behave worse than in the massless case. This is understandable if one compares the coefficients of the logarithms in the spectral densities (18) and (21). The coefficients are larger overall in the mass correction case. The basic objects one needs for the construction of observables are moments of the spectral density $\rho_g(s)$

$$M_g(n) = (n+1)\int_0^1 \rho_g(s) s^n ds \ . \tag{23}$$

We find

$$\begin{aligned}
M_g(0) &= 1 + 2a + 16.6a^2 + 137.0a^3 + (1378.5 - 2.0k_3)a^4 \\
M_g(1) &= 1 + a + 6.15a^2 + 28.67a^3 + (141.97 - 1.0k_3)a^4 \\
M_g(2) &= 1 + \frac{2}{3}a + 3.63a^2 + 12.31a^3 + (35.69 - 0.67k_3)a^4 \\
M_g(3) &= 1 + \frac{1}{2}a + 2.54a^2 + 6.97a^3 + (11.58 - 0.5k_3)a^4 \\
M_g(4) &= 1 + \frac{2}{5}a + 1.95a^2 + 4.56a^3 + (3.55 - 0.4k_3)a^4 \\
&\ \ \vdots \\
M_g(100) &= 1 + \frac{2}{101}a + 0.081a^2 + 0.060a^3 + (-0.434 - 0.0198k_3)a^4
\end{aligned} \tag{24}$$

Note that the unknown coefficient k_{g3}, which would appear in the fourth order coefficient of the moments in the $\overline{\mathrm{MS}}$-scheme, is absorbed in the definition of the mass m_g. Still the fourth order coefficient is not known because of its dependence on k_3 which enters due to the charge redefinition. Indeed, this dependence has its origin in the definition of the mass m_g (19). The third coefficient of the effective γ-function γ_{g3} depends on k_3. This dependence affects the fourth order coefficient of $\rho_g(s)$ through the running mass (11)

$$\gamma_0 = 1, \quad \gamma_{g1} = 4.027, \quad \gamma_{g2} = 2.65, \quad \gamma_{g3} = -22.65 - k_3 \, . \tag{25}$$

For large n the moments behave better because the infra-red region of integration is sup-

pressed. Note that the coefficients of the series in eq. (24) are saturated with the lowest power of logarithm for large n for a given order of perturbation theory, i.e. they are saturated with the highest coefficient of the effective β-function and γ-function.

It is instructive to compare the results in eq. (24) with the $\overline{\rm MS}$-scheme expansions given by

$$\begin{aligned}
M_g^{\overline{\rm MS}}(0) &= 1+3.67\frac{\alpha_s}{\pi}+20.0\left(\frac{\alpha_s}{\pi}\right)^2+110.1\left(\frac{\alpha_s}{\pi}\right)^3+(-256.3+k_{g3})\left(\frac{\alpha_s}{\pi}\right)^4\\
M_g^{\overline{\rm MS}}(1) &= 1+2.67\frac{\alpha_s}{\pi}+6.32\left(\frac{\alpha_s}{\pi}\right)^2-38.98\left(\frac{\alpha_s}{\pi}\right)^3+(-1779+k_{g3})\left(\frac{\alpha_s}{\pi}\right)^4\\
M_g^{\overline{\rm MS}}(2) &= 1+2.33\frac{\alpha_s}{\pi}+2.70\left(\frac{\alpha_s}{\pi}\right)^2-64.26\left(\frac{\alpha_s}{\pi}\right)^3+(-1865+k_{g3})\left(\frac{\alpha_s}{\pi}\right)^4\\
M_g^{\overline{\rm MS}}(3) &= 1+2.17\frac{\alpha_s}{\pi}+1.06\left(\frac{\alpha_s}{\pi}\right)^2-73.18\left(\frac{\alpha_s}{\pi}\right)^3+(-1863+k_{g3})\left(\frac{\alpha_s}{\pi}\right)^4\\
M_g^{\overline{\rm MS}}(4) &= 1+2.07\frac{\alpha_s}{\pi}+0.14\left(\frac{\alpha_s}{\pi}\right)^2-77.46\left(\frac{\alpha_s}{\pi}\right)^3+(-1851+k_{g3})\left(\frac{\alpha_s}{\pi}\right)^4\\
&\ \vdots\\
M_g^{\overline{\rm MS}}(100) &= 1+1.69\frac{\alpha_s}{\pi}+2.99\left(\frac{\alpha_s}{\pi}\right)^2-87.15\left(\frac{\alpha_s}{\pi}\right)^3+(-1755+k_{g3})\left(\frac{\alpha_s}{\pi}\right)^4 .
\end{aligned}$$

The advantage of the effective scheme against the $\overline{\rm MS}$-scheme is apparent starting with α_s^3 coefficient. For moments larger than one ($n>1$) in the $\overline{\rm MS}$-scheme the series shows already asymptotic growth in the third order while in the effective scheme the third coefficient is still smaller than the previous one. From its construction it is clear that the convergence behavior improves with higher moments in the effective scheme while in $\overline{\rm MS}$-scheme high moments become even worse than the lower ones from the point of view of the structure of the perturbation theory series. The anomalously small third order coefficient of the first moment in the $\overline{\rm MS}$-scheme is the result of an accidental cancellation of the contributions of logarithmic and constant terms because of the particular choice of the scheme. The general discussion of ref. [1] now applies. For our numerical estimates we take $a=0.111$ as obtained from the corresponding value of the $\overline{\rm MS}$-scheme charge. In the effective scheme the PT series

for the moments read

$$
\begin{aligned}
M_g(0) &= 1 + 0.222 + 0.204 + 0.187 + (0.21 - 0.0003k_3) \\
M_g(1) &= 1 + 0.111 + 0.076 + 0.039 + (0.022 - 0.00015k_3) \\
M_g(2) &= 1 + 0.074 + 0.045 + 0.017 + (0.0054 - 0.00010k_3) \\
M_g(3) &= 1 + 0.056 + 0.031 + 0.010 + (0.0018 - 0.000076k_3) \\
M_g(4) &= 1 + 0.044 + 0.024 + 0.006 + (0.00054 - 0.000061k_3)\,.
\end{aligned}
\tag{26}
$$

With the choice $k_3 = 100$ [1] the fourth order correction is smaller than the third term for the moments with $n < 5$. A formal accuracy of about 0.7% can be obtained if the zero order moment is excluded using as an estimate the contribution of the smallest term. With the standard Padé estimate for $k_3 = 25$ the moments become

$$
\begin{aligned}
M_g(0) &= 1 + 0.222 + 0.204 + 0.187 + 0.20 \\
M_g(1) &= 1 + 0.111 + 0.076 + 0.039 + 0.018 \\
M_g(2) &= 1 + 0.074 + 0.045 + 0.017 + 0.003 \\
M_g(3) &= 1 + 0.056 + 0.031 + 0.010 - 0.0001 \\
M_g(4) &= 1 + 0.044 + 0.024 + 0.006 - 0.001\,.
\end{aligned}
\tag{27}
$$

If one excludes $M_g(0)$ an accuracy of better than 2% can be obtained.

There is no value for k_3 which makes the forth order corrections of all moments smaller than the previous correction. If k_3 has a value between -46.5 and 1.2 all moments starting from the first moment show no asymptotic growth in fourth order. Such a fine tuning of the unknown coefficient k_3 seems to be unrealistic. We thus conclude that asymptotic growth is unavoidable in fourth order in the g-part. The ultimate accuracy depends on the value of k_3 varying between 0.7% and 2% if the zero order moment is excluded. The invariant statement about the asymptotic growth is that the system of moments $M_g(n)$ with $n = 0$ included cannot be treated perturbatively at the fourth order of perturbation theory for

the given numerical value of the expansion parameter $a = 0.111$ if one wants to obtain an accuracy of the coefficient function in front of m_s^2 correction in $\Pi_g(q^2)$ amplitude better than 15% - 20%. This statement about the ultimate accuracy of the set of moment observables attainable in fourth order of perturbation theory is independent of whichever numerical value k_3 takes.

The perturbation theory expansions for the system of moments with $(1-s)^n$ weight

$$\tilde{M}_g(n,0) = (n+1)\int_0^1 \rho_g(s)(1-s)^n ds = (n+1)!\sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!(n-k)!} M_g(k) \qquad (28)$$

show a much worse PT behavior. One has

$$\begin{aligned}
\tilde{M}_g(1,0) &= 1 + 0.333 + 0.332 + 0.336 + (0.397 - 0.00046k_3) \\
\tilde{M}_g(2,0) &= 1 + 0.407 + 0.429 + 0.461 + (0.569 - 0.00056k_3) \\
\tilde{M}_g(3,0) &= 1 + 0.463 + 0.509 + 0.572 + (0.728 - 0.00063k_3)\,. \qquad (29)
\end{aligned}$$

The convergence of the series is obviously quite poor. All moments $\tilde{M}_g(n,0)$ contain the contribution of $\tilde{M}_g(0,0) \equiv M_g(0)$ in the sum eq. (28) which by itself shows a bad behavior; the rest makes it worse.

To summarize, the general PT structure of the moments for the g-part in the effective scheme with the new mass parameter m_g^2 is very similar to the massless part and there are no qualitatively new features found here as compared to the analysis of the massless part in ref. [1].

2.2 The q-part of the correlator: function $\Pi_{mq}(q^2)$

The q-amplitude $\Pi_q(q^2)$ contains contributions of both spin 1 and spin 0 final states. The correction to the q-part is different from the g-part as concerns its analytic properties in the q^2-plane – it contains a $1/q^2$ singularity at the origin – which necessitates a separate treatment. The explicit power singularity $1/q^2$ at the origin of the function $\Pi_{mq}(q^2)/q^2$

makes the formulation of the moments for the spectral density of the q-part directly on the physical cut a bit tricky. Indeed, because of the $1/q^2$ factor the amplitude for the m_s^2 correction has no standard dispersion representation. Rather the dispersion representation should be written as

$$\frac{\Pi_{mq}(Q^2)}{Q^2} = \int \frac{d\sigma(s)}{s+Q^2} \tag{30}$$

with a measure $d\sigma(s)$ which is not differentiable, i.e. $d\sigma(s) \neq \sigma'(s)ds$ with some continuous $\sigma'(s)$. However, it can be written in a more familiar form if a different weight is used

$$\frac{\Pi_{mq}(Q^2)}{Q^2} = \int_0^\infty \frac{\rho_F(s)ds}{(s+Q^2)^2} = -\frac{d}{dQ^2}F(Q^2) \tag{31}$$

with

$$F(Q^2) = \int_0^\infty \frac{\rho_F(s)ds}{s+Q^2} \tag{32}$$

and $\rho_F(s)$ a continuous spectral density. Therefore $F(Q^2)$ is the primary function of

$$-\Pi_{mq}(Q^2)/Q^2 = \Pi_{mq}(q^2)/q^2\,.$$

It reads

$$F(Q^2) = -\int \frac{dQ^2}{Q^2}\Pi_{mq}(Q^2) = \int dL\;\Pi_{mq}(Q^2) \quad L = \ln\frac{\mu^2}{Q^2}\,. \tag{33}$$

The result for $F(Q^2)$ is

$$F(Q^2) = L + (\frac{7}{3}L + L^2)a + (15.757L + 7.11L^2 + 1.417L^3)a^2 + ... \tag{34}$$

For the discontinuity across the cut defined by

$$\rho_F(s) = \frac{1}{2\pi i}\left(F(-s-i0) - F(-s+i0)\right) \tag{35}$$

one obtains

$$\begin{aligned}\rho_F(s) \quad = \quad & 1 + (\frac{7}{3} + 2l)a + (1.77 + 14.22l + 4.25l^2)a^2 + \\ & (-207.04 + k_{q2} + 62.21l + 54.52l^2 + 9.21l^3)a^3 + ...\end{aligned} \tag{36}$$

where we have already substituted the effective coupling a for α_s. Eq. (36) has the standard form of the spectral density for needed for the comparison with the massless and the g cases. In order to get rid of the constants the new mass parameter m_q for the amplitude $\Pi_{mq}(Q^2)$ is defined in analogy with the g-case such that

$$m_q^2 = m_s^2 \rho_F(M_\tau^2) \tag{37}$$

with $m_q^2 = m_q^2(M_\tau^2)$ and $m_s^2 = m_s^2(M_\tau^2)$. Then one explicitly has

$$m_q^2 = m_s^2(1+\frac{7}{3}a+1.77a^2+(-207.044+k_{q2})a^3+(-1335.5-2.33k_3-4.92k_{q2}+k_{q3})a^4+...) \tag{38}$$

Note that the order a^4 term contains not only the unknown coefficient k_{q2} but also the higher order coefficient k_{q3} which makes the contribution of the a^4 term completely arbitrary. The definition of this new mass m_q^2 in terms of the $\overline{\mathrm{MS}}$-scheme mass eq. (38) has an accuracy of about 2% if only second order corrections are used. The PT series is only known to third order – already the fourth term contains the unknown coefficient k_{q2}. Expressed through the $\overline{\mathrm{MS}}$-scheme coupling constant α_s eq. (38) reads

$$m_q^2 = m_s^2 \left(1 + \frac{7}{3}\left(\frac{\alpha_s}{\pi}\right) + 5.60\left(\frac{\alpha_s}{\pi}\right)^2 + (-225.22 + k_{q2})\left(\frac{\alpha_s}{\pi}\right)^3 + ...\right) \tag{39}$$

With this new mass m_q^2 a new spectral density $\rho_q(s)$ can be defined in analogy to eq. (19)

$$m_s^2 \rho_F(s) = m_q^2 \rho_q(s) \tag{40}$$

that leads to the expansion

$$\begin{aligned} \rho_q(s) \;&=\; 1 + 2la + (9.55l + 4.25l^2)a^2 + (36.36l + 44.6l^2 + 9.21l^3)a^3 + \\ &\quad ((-1141 - 2k_3 + 6.75k_{q2})l + 253.6l^2 + 154.96l^3 + 20.14l^4)a^4 + O(a^5)\,. \end{aligned} \tag{41}$$

Note that the series (41) contains only logarithms and no constants. The coefficients of the spectral density $\rho_q(s)$ from eq. (41) are close to those of the g-part mass correction $\rho_g(s)$ eq. (21) up to the second order. The coefficients of the highest powers of logarithms

are simply equal while the lower powers are different because of different coefficients in the corresponding D-functions eq. (9). The coefficient of the third order term of eq. (41) for $\rho_q(s)$ is seven times larger than the corresponding coefficient from eq. (21) for $\rho_g(s)$ for the lowest power of logarithm which dominates the behavior of the higher moments. The coefficients of the spectral density $\rho_q(s)$ are in general larger than the coefficients of the spectral density $\rho(s)$ in the massless case eq. (18). Thus, the behavior of the spectral densities (ρ, ρ_g, ρ_q) allows one to immediately conclude about the convergence of the corresponding moments for all three independent contributions up to order m_s^2 in the strange decays.

The standard moments of $\rho_q(s)$ as in eq. (13,23) do not however coincide with the physical moments of the q-amplitude. Indeed, the physical q-moments $M_q^{ph}(n)$ are defined through the contour integration in the following way

$$\frac{im_s^2}{2\pi}\oint \frac{\Pi_{mq}(q^2)}{q^2}\left(\frac{q^2}{M_\tau^2}\right)^n dq^2 = m_q^2 M_q^{ph}(n), \tag{42}$$

where the effective mass m_q^2 is used for normalization. The moments eq. (42) can be evaluated through the function $\rho_q(s)$ directly. The zero order moment turns out to be

$$M_q^{ph}(0) = -\rho_q(M_\tau^2) = -1\,. \tag{43}$$

The reason for this is that the integration with $n = 0$ in eq. (42) picks up exactly those contributions that eventually are absorbed into the strange quark mass redefinition. Higher order physical moments are related to the standard moments of $\rho_q(s)$ via

$$M_q^{ph}(n)|_{n>0} = n\int_0^1 \rho_q(s)s^{n-1}ds - 1 \equiv M_q(n-1) - 1\,. \tag{44}$$

They contain no parton model contribution. The moments $M_q(n)$ are the standard objects defined in eq. (13) and the whole analysis of ref. [1] is applicable with $\rho_q(s)$ to be compared with $\rho_g(s)$ and $\rho(s)$. Numerical values for the moments are

$$\begin{aligned} M_q(0) &= 1 + 2a + 18.1a^2 + 180.8a^3 + (779.4 - 2k_3 + 6.75k_{q2})a^4 \\ M_q(1) &= 1 + 1a + 6.90a^2 + 47.39a^3 + (-297.3 - k_3 + 3.375k_{q2})a^4 \end{aligned}$$

$$
\begin{aligned}
M_q(2) &= 1+\frac{2}{3}a+4.13a^2+24.08a^3+(-283.6-0.67k_3+2.25k_{q2})a^4 \\
M_q(3) &= 1+\frac{1}{2}a+2.92a^2+15.53a^3+(-237.13-0.5k_3+1.69k_{q2})a^4 \\
M_q(4) &= 1+\frac{2}{5}a+2.25a^2+11.28a^3+(-199.699-0.4k_3+1.35k_{q2})a^4 \\
&\vdots \\
M_q(100) &= 1+\frac{1}{50}a+0.095a^2+0.37a^3+(-11.25-0.020k_3+0.067k_{q2})a^4\,. \qquad (45)
\end{aligned}
$$

The use of $\rho_q(s)$ is a universal way of describing the moments in FOPT and easy for comparison with the massless and the g parts. The physical q-moments $M_q^{ph}(n)$ are related to it by eq. (43) and (44). They do not, however, change the pattern of PT convergence. The moment $M_q^{ph}(1)$ related to $M_q(0)$ shows asymptotic growth in the third order already. As expected higher moments show a better behavior because the low energy region of the integration is suppressed.

3 Order m_s^2 corrections to τ-lepton decay observables

After introducing the technique for analyzing the moments at order m_s^2 we now apply it to an analysis of physical observables.

3.1 Total decay rate

The τ decay width is given by a specific linear combination of moments. The weight function contains the overall factor $(1-s)^2$ which impairs the convergence of the total decay rate observable. The $(1-s)^2$ factor enhances the infra-red region of integration, i.e. the relative magnitude of the contributions of logarithms $\ln(M_\tau^2/s)$ at small energy. The concrete shape of the weight function with the weight factor $(1-s)^2$ is the main source of slow convergence of the m_s^2 correction to the rate

$$
R_{m\tau}=\frac{i}{2\pi}\oint 2\left(1-\frac{q^2}{M_\tau^2}\right)^2 3\left(\frac{m_s^2\Pi_{mq}(q^2)}{q^2}-\frac{m_s^2}{M_\tau^2}\Pi_{mg}(q^2)\right)\frac{dq^2}{M_\tau^2}\,. \qquad (46)
$$

We retain the different mass definitions for the two invariant functions in order to explore the structure of the PT series and to check on its convergence.

The result for the mass correction of the total decay rate reads

$$\begin{aligned} R_{m\tau} &= 6\frac{m_q^2}{M_\tau^2}(M_q^{ph}(0) - 2M_q^{ph}(1) + M_q^{ph}(2)) - 6\frac{m_g^2}{M_\tau^2}(M_g(0) - M_g(1) + \frac{1}{3}M_g(2)) \\ &= -6\frac{m_q^2}{M_\tau^2}(2M_q(0) - M_q(1)) - 6\frac{m_g^2}{M_\tau^2}(M_g(0) - M_g(1) + \frac{1}{3}M_g(2)) \,. \end{aligned} \tag{47}$$

It is expressed through the effective mass parameters $m_{g,q}^2$ and the moments introduced earlier. In the parton model approximation all the moments are normalized to unity which makes a glance analysis of eq. (47) easy. The convergence pattern of all the moments has been obtained already before. Numerically one has

$$\begin{aligned} R_{m\tau} &= -6\frac{m_q^2}{M_\tau^2}\left(1 + 3a + 29.21a^2 + 314.3a^3 + (1856.1 - 3.0k_3 + 10.13k_{q2})a^4\right) \\ &\quad -2\frac{m_g^2}{M_\tau^2}\left(1 + 3.67a + 34.84a^2 + 337.3a^3 + (3745.2 - 3.67k_3)a^4\right) \end{aligned} \tag{48}$$

The reason for the bad convergence of (48) is the contribution of the low moments $M_q(0)$, $M_q(1)$ and $M_g(0)$, $M_g(1)$ to the mass correction of the total decay rate. Both series in eq. (48) converge only marginally calling for the resummation of the series. The total contribution is dominated by the q-part which has a three times bigger coefficient in the leading term.

Inserting the expressions for m_q^2, m_g^2, a in terms of the $\overline{\rm MS}$-scheme parameters m_s^2 and α_s into eq. (48) one obtains the standard result

$$R_{m\tau} = -8\frac{m_s^2}{M_\tau^2}\left(1 + 5.33\frac{\alpha_s}{\pi} + 46.0\left(\frac{\alpha_s}{\pi}\right)^2 + (283.55 + 0.75k_{q2})\left(\frac{\alpha_s}{\pi}\right)^3 + ...\right) . \tag{49}$$

3.2 The '1+0' Method

In ref. [2] the numerical value for the strange quark mass in $\overline{\rm MS}$-scheme has been extracted with the '1+0' method which uses the representation of the total decay rate as a sum of $(L+T)$ and L contributions (compare with eq. (6))

$$R_\tau = \frac{i}{2\pi}\oint 2\left(1 - \frac{q^2}{M_\tau^2}\right)^2 \left\{\left(1 + 2\frac{q^2}{M_\tau^2}\right)\Pi_{(L+T)}(q^2) - 2\frac{q^2}{M_\tau^2}\Pi_L(q^2)\right\}\frac{dq^2}{M_\tau^2} \,, \tag{50}$$

$$\Pi_{(L+T)}(q^2) = \Pi_q(q^2)\,. \tag{51}$$

It is assumed that in the $\overline{\rm MS}$-scheme the series for the $(L+T)$ part of the m_s^2 correction in eq. (50) converges well [2]. The convergence is not impressive in FOPT but the numbers given for contour-improved FOPT in ref. [2] show fast convergence. The quantity of interest is now

$$\begin{aligned} R_{m\tau}^{L+T} &= \frac{i}{2\pi}\oint 2\left(1-\frac{q^2}{M_\tau^2}\right)^2\left(1+2\frac{q^2}{M_\tau^2}\right)3\frac{\Pi_{mq}(q^2)}{q^2}\frac{dq^2}{M_\tau^2} \\ &= 6\frac{m_q^2}{M_\tau^2}(M_q^{ph}(0)-3M_q^{ph}(2)+2M_q^{ph}(3)) \\ &= -6\frac{m_q^2}{M_\tau^2}(3M_q(1)-2M_q(2)). \end{aligned} \tag{52}$$

If we look at the moments which are contained in $R_{m\tau}^{L+T}$ (52) it is natural to expect a convergence behavior better than in the total decay rate because no zero order moments are needed to construct $R_{m\tau}^{L+T}$. This is an invariant (scheme independent) reason for better convergence: the particular combination $R_{m\tau}^{L+T}$ receives smaller IR contribution from integration along the cut and therefore is better computable in PT. Still the convergence is rather slow. Numerical results for the $(L+T)$ part are

$$R_{m\tau}^{L+T}|_{q-scheme} =$$
$$-6\frac{m_q^2}{M_\tau^2}\left(1+1.67a+12.448a^2+94.01a^3+(-324.629-1.67k_3+5.625k_{q2})a^4+\ldots\right)\,. \tag{53}$$

The convergence persists and the last term is still smaller than the third for the standard values of $25 < k_3 < 100$ and $0 < k_{q2} < 160$. The total contribution of first three terms is 0.45 which is a reasonable change of the leading order term due to PT correction. In the $\overline{\rm MS}$-scheme this becomes

$$\begin{aligned} R_{m\tau}^{L+T}|_{\overline{\rm MS}-scheme} &= -6\frac{m_s^2}{M_\tau^2}\left(1+4.0\left(\frac{\alpha_s}{\pi}\right)+24.67\left(\frac{\alpha_s}{\pi}\right)^2+(-62.77+k_{q2})\left(\frac{\alpha_s}{\pi}\right)^3+\right. \\ &\left.(-3110+7.29k_{q2}+k_{q3})\left(\frac{\alpha_s}{\pi}\right)^4+...\right) \end{aligned} \tag{54}$$

The total contribution of the first two terms amounts to 0.65. The change in the leading order prediction is even larger than the total change in the effective q-scheme where one more term of PT expansion is available. The only advantage of the '1+0' amplitude from PT point of view is the absence of the zero order moment $M_q(0)$ which is the most divergent one. Still, the moment $M_q(1)$ from eq. (45) which contributes to the rate expression eq. (52) is bad enough to prevent a fast convergence. The longitudinal part can be expressed in terms of the moments and convergence is rather bad

$$\begin{aligned} R^L_{m\tau} &= \frac{i}{2\pi}\oint 2\left(1-\frac{q^2}{M_\tau^2}\right)^2(-2)\frac{q^2}{M_\tau^2}\Pi_L(q^2)\frac{dq^2}{M_\tau^2} \\ &= -12\frac{m_q^2}{M_\tau^2}(M_q(0)-2M_q(1)+M_q(2))-6\frac{m_g^2}{M_\tau^2}(M_g(0)-M_g(1)+\frac{1}{3}M_g(2)) \end{aligned} \quad (55)$$

because again the lowest order moments enter. The leading order parton model term in the q-part vanishes. Numerical values are

$$\begin{aligned} R^L_{m\tau} = &- 2\frac{m_g^2}{M_\tau^2}\left(1+3.67a+34.8a^2+337.3a^3+(3745.2-3.67k_3)a^4\right) \\ &- 8\frac{m_q^2}{M_\tau^2}\left(0+a+12.57a^2+165.2a^3+(1635.6-k_3+3.375k_{q2})a^4\right) \end{aligned} \quad (56)$$

from which one can perceive the reason for the bad PT structure. While the convergence of the g-part is rather standard, the admixture of q-contribution without the leading term is enhanced by relative factor of four which makes the sum in eq. (56) completely uninterpretable.

Note that we use $m_{g,q}$ mass parameters for g, q parts of the correlator as internal mass scales for m_s^2 corrections. One can introduce another set of parameters $m_{T,L}$ related to the definite spin decomposition of the correlator eq. (4). As one can see from eq. (5) $m_T = m_g$ because the $\Pi_T(q^2)$ amplitude is proportional to $\Pi_g(q^2)$. The effective mass m_L^2 is obtained from th longitudinal part $\Pi_L(q^2)$. The expression of m_T^2 through the $\overline{\rm MS}$-scheme mass is reasonable (eq. (20)) while the corresponding relation for m_L is much wilder.

$$m_L^2 = m_s^2\left(1+5.67\frac{\alpha_s}{\pi}+31.9\left(\frac{\alpha_s}{\pi}\right)^2+89.2\left(\frac{\alpha_s}{\pi}\right)^3+\right.$$
$$\left.(-5180+k_{g3}+17.5k_{q2})\left(\frac{\alpha_s}{\pi}\right)^4+...\right) \tag{57}$$

The different structures of PT serieses for the parameters m_T^2 ans m_L^2 can be interpreted as a result of the difference of full QCD interaction in spin 1 and spin 0 channels. The stronger and earlier breakdown of PT behavior in the spin 0 channels can be related to the contribution of non-perturbative effects (instantons) which is absent in spin 1 channel.

To summarize, the convergence of the PT series for the m_s^2 corrections for the most natural and precisely measured physical observables is always slow and almost marginal. This is because the IR region of integration is numerically important for the given value of the coupling constant and the order of PT. Any apparent fast convergence is the result of either a specific linear combination of moments or a particular scheme choice. The former case is, however, not realized for physical observables of interest measurable in experiment. This would imply that a resummation of the series is necessary for a sound interpretation of the theoretical formulas for decay rates.

4 Conclusions

We have analyzed the asymptotic structure of PT series for m_s^2 corrections. Using the standard estimate of the accuracy of an asymptotic series we have found that the theoretical precision in the perturbative description of Cabibbo supressed τ-lepton decays is already limited by the asymptotic growth of the coefficients in fourth order of perturbation theory. This is a scheme invariant statement. The accuracy of the perturbative expansion for the coefficient functions of the m_s^2 corrections in Cabibbo suppressed channels are 15%-20% at best. Therefore the extraction of the numerical value for the strange quark mass from the m_s^2 corrections to the τ-decay rate into strange hadrons is limited by the precision of

the coefficient functions. Better theoretical accuracy can be obtained by using observables which contain higher order moments but the experimental accuracy of them is not good enough at present. From phenomenological point of view the analysis of the m_s^2 corrections differ from the massless case. While in the latter the low order moments can be excluded by substituting experimantal results for them (from e^+e^- annihilation, for instance), the coefficient functions of m_s^2 corrections have no immediate physical meaning and cannot be traded for in this manner.

The introduction of two natural mass parameters allows to describe massless, q and g parts of the correlator up to order a^4 with only two unknown parameters k_3 and k_{q2} instead of four in the $\overline{\mathrm{MS}}$-scheme. The existence of two different mass scales is physically motivated by the difference of interaction in spin 1 and spin 0 channels. Still for both independent m_s^2 corrections (g and q part) the convergence of low order moments is slow. The contribution of IR region is large and these quantities have limited precision when evaluated within PT. It can not be improved by a particular change of the scheme or taking particular linear combinations. Only those moments converge well where the IR contribution is suppressed. The renormalization group improved QCD parameters $a(s)$, $m_s(s)$ run too fast to give precise PT series for a set of τ observables containing low moments with large IR contribution. If the running would be slower – coefficients of β- and γ-functions would be smaller – then one could meet the high standards of experimental precision for low moments with FOPT theoretical formulas. Observables with higher moments are described well within FOPT but experimental accuracy is not yet sufficient for precise comparison.

Therefore for precise comparison of theory with experiment some procedure of resummation is required [15, 21, 24]. And, in fact, one can perform resummation only for lowest order moments to keep things close to the standard FOPT. Resumation, however, introduces additional (to the standard renormalization group freedom [25]) arbitrariness, in particular, it interferes with nonperturbative power corrections that makes separation of contributions

within the OPE not unique.

This problem is beyond the scope of the present analysis.

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Appendix

The input for all calculations are the coefficients of the correlator, the beta and the gamma function which are available up to four loop calculations in present. In the case of $\Pi_q(Q^2)$ only corrections up to α_s^2 are known because the constant term of the correlator is not yet calculable. All coefficients are given in the $\overline{\text{MS}}$-scheme (see, e.g. [26]). Below $\zeta(z)$ is the Riemann ζ-function.

The coefficients of the correlator are [27, 28]

$$
\begin{aligned}
k_1 &= \frac{299}{24} - 9\zeta(3), \\
k_2 &= \frac{58057}{288} - \frac{779}{4}\zeta(3) + \frac{75}{2}\zeta(5), \\
k_{q1} &= \frac{13981}{432} + \frac{323}{54}\zeta(3) - \frac{520}{27}\zeta(5), \\
k_{g1} &= \frac{4591}{144} - \frac{35}{2}\zeta(3), \\
k_{g2} &= \frac{1967833}{5184} - \frac{\pi^4}{36} - \frac{11795}{24}\zeta(3) + \frac{33475}{108}\zeta(5)\,.
\end{aligned}
$$

Beta function coefficients [29]

$$
\begin{aligned}
\beta_0 &= \frac{9}{4}, \quad \beta_1 = 4, \quad \beta_2 = \frac{3863}{384}, \\
\beta_3 &= \frac{140599}{4608} + \frac{445}{32}\zeta(3)\,.
\end{aligned}
$$

Gamma function coefficients [30, 31]

$$
\begin{aligned}
\gamma_0 &= 1, \quad \gamma_1 = \frac{91}{24}, \quad \gamma_2 = \frac{8885}{578} - \frac{5}{2}\zeta(3), \\
\gamma_3 &= \frac{2977517}{41472} + \frac{3\pi^4}{32} - \frac{9295}{432}\zeta(3) - \frac{125}{12}\zeta(5)\,.
\end{aligned}
$$